10 Pasch Geometries

Definition (Pasch's Postulate (PP))
A metric geometry satisfies Pasch's Postulate (PP) if for any line ℓ, any triangle $\triangle ABC$, and any point $D \in \ell$ such that $A - D - B$, then either $\ell \cap \overline{AC} \neq \emptyset$ or $\ell \cap \overline{BC} \neq \emptyset$.

Theorem (Pasch's Theorem) If a metric geometry satisfies PSA then it also satisfies PP.

1. Prove the above theorem.

Definition (Pasch Geometry)
A Pasch Geometry is a metric geometry which satisfies PSA.

Theorem Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry which satisfies PP. If A, B, C are noncollinear and if the line ℓ does not contain any of the points A, B, C, then ℓ cannot intersect all three sides of $\triangle ABC$.

2. Prove the above theorem.

Theorem If a metric geometry satisfies PP then it also satisfies PSA.

3. Prove the above theorem.

4. (Peano's Axiom) Given a triangle $\triangle ABC$ in a metric geometry which satisfies PSA and points D, E with $B - C - D$ and $A - E - C$, prove there is a point $F \in \overleftrightarrow{DE}$ with $A - F - B$, and $D - E - F$.

5. Given $\triangle ABC$ in a metric geometry which satisfies PSA and points D, F with $B - C - D$, $A - F - B$, prove there exists $E \in \overleftrightarrow{DF}$ with $A - E - C$ and $D - E - F$.

6. Given $\triangle ABC$ and a point P in a metric geometry which satisfies PSA prove there is a line through P that contains exactly two points of $\triangle ABC$.

Definition (Missing Strip Plane)
The Missing Strip Plane is the abstract geometry $\{\mathcal{S}, \mathcal{L}\}$ given by
$\mathcal{S} = \{(x, y) \in \mathbb{R}^2 \mid x < 0 \text{ or } 1 \leq x\}$,
$\mathcal{L} = \{\ell \cap \mathcal{S} \mid \ell \text{ is a Cartesian line and } \ell \cap \mathcal{S} \neq \emptyset\}$.

7. Given the following pairs of points: (i) $(2, 3)$ and $(3, -1)$; (ii) $(0, 3)$ and $(1/2, -2)$; (iii) $(-1, 4)$ and $(2, 7)$. If the given pair of points lies in the point set of the Missing Strip Plane, find the line through that pair of points.

8. If lines ℓ_1, ℓ_2 and ℓ_3 in the Missing Strip plane satisfy:
ℓ_1 is parallel to ℓ_2 and
ℓ_2 is parallel to ℓ_3,
is it true that ℓ_1 is parallel to ℓ_3? Justify your answer.

9. Given that a metric geometry satisfies PSA if and only if it is a Pasch geometry, give an example to show that the Missing Strip Plane does not satisfy PSA.

10. Let $\mathcal{S}$ denote the set of points of the Missing Strip plane. Find all lines in this plane through the point $(2, 0)$ which are parallel in the Missing Strip plane to (i) the line $L_{-1} \cap \mathcal{S}$; (ii) the line $L_{1,2} \cap \mathcal{S}$.

11. Prove that the Missing Strip Plane is an incidence geometry.

Proposition If $\{\mathcal{S}, \mathcal{L}\}$ is the Missing Strip Plane and $\ell = L_{m,b}$ then $g_\ell : \ell \cap \mathcal{S} \to \mathbb{R}$ is a bijection (for definition of g_ℓ see lecture notes or in book on page 79).

12. Prove the above proposition.

Proposition The Missing Strip Plane is not a Pasch geometry.

13. Prove the above proposition.

14. Let $\mathcal{S}$ denote the set of points of the Missing Strip plane. Find all lines in this plane through the point $(-1, 1)$ which are parallel in the Missing Strip plane to (i) the line $L_2 \cap \mathcal{S}$; (ii) the line $L_{-1,2} \cap \mathcal{S}$.

15. Given a triangle, $\triangle ABC$, in a metric geometry, and points D, E with $A - D - B$ and $C - E - B$, is it always the case that $\overleftrightarrow{AE} \cap \overleftrightarrow{CD} \neq \emptyset$?

11 Interiors and the Crossbar Theorem

Theorem In a Pasch geometry if $\mathcal{A}$ is a non-empty convex set that does not intersect the line ℓ, then all points of $\mathcal{A}$ lie on the same side of ℓ.

1. Prove the above theorem.

Definition (interior of the ray, interior of the segment)

The interior of the ray $\overrightarrow{AB}$ in a metric geometry is the set $\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}$. The interior of the segment $\overline{AB}$ in a metric geometry is the set $\text{int}(\overline{AB}) = \overline{AB} - \{A, B\}$.

2. Prove that in a metric geometry, $\text{int}(\overrightarrow{AB})$ and $\text{int}(\overline{AB})$ are convex sets.

Theorem Let $\mathcal{A}$ be a line, ray, segment, the interior of a ray, or the interior of a segment in a Pasch geometry. If ℓ is a line with $\mathcal{A} \cap \ell = \emptyset$ then all of $\mathcal{A}$ lies on one side of ℓ. If there is a point B with $A - B - C$ and $\overleftrightarrow{AC} \cap \ell = \{B\}$ then $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overline{BA})$ both lie on the same side of ℓ while $\text{int}(\overrightarrow{BA})$ and $\text{int}(\overrightarrow{BC})$ lie on opposite sides of ℓ.

3. Prove the above theorem.

Theorem (Z Theorem) In a Pasch geometry, if P and Q are on opposite sides of the line $\overleftrightarrow{AB}$ then $\overrightarrow{BP} \cap \overrightarrow{AQ} = \emptyset$. In particular, $\overline{BP} \cap \overline{AQ} = \emptyset$.

4. Prove the above theorem.

Definition (interior of $\measuredangle ABC$)

In a Pasch geometry the interior of $\measuredangle ABC$, written $\text{int}(\measuredangle ABC)$, is the intersection of the side of $\overleftrightarrow{AB}$ that contains C with the side of $\overleftrightarrow{BC}$ that contains A.

Theorem In a Pasch geometry, if $\measuredangle ABC = \measuredangle A'B'C'$ then $\text{int}(\measuredangle ABC) = \text{int}(\measuredangle A'B'C')$.

5. Prove the above theorem.

Theorem In a Pasch geometry, $P \in \text{int}(\measuredangle ABC)$ if and only if A and P are on the same side of $\overleftrightarrow{BC}$ and C and P are on the same side of $\overleftrightarrow{BA}$.

6. Prove the above theorem.

Theorem Given $\triangle ABC$ in a Pasch geometry, if $A - P - C$ then $P \in \text{int}(\measuredangle ABC)$ and therefore $\text{int}(\overline{AC}) \subseteq \text{int}(\measuredangle ABC)$.

7. Prove the above theorem.

8. In a Pasch geometry, if $P \in \text{int}(\measuredangle ABC)$ prove $\text{int}(\overrightarrow{BP}) \subseteq \text{int}(\measuredangle ABC)$.

Theorem (Crossbar Theorem) In a Pasch geometry if $P \in \text{int}(\measuredangle ABC)$ then $\overrightarrow{BP}$ intersects $\overline{AC}$ at a unique point F with $A - F - C$.

9. Prove the above theorem.

Theorem In a Pasch geometry, if $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$ then $P \in \text{int}(\measuredangle ABC)$ if and only if A and C are on opposite sides of $\overleftrightarrow{BP}$.

10. Prove the above theorem.

Theorem In a Pasch geometry, if $A - B - D$ then $P \in \text{int}(\measuredangle ABC)$ if and only if $C \in \text{int}(\measuredangle DBP)$.

11. Prove the above theorem.

Definition (interior of $\triangle ABC$)

In a Pasch geometry, the interior of $\triangle ABC$, written $\text{int}(\triangle ABC)$, is the intersection of the side of $\overleftrightarrow{AB}$ which contains C, the side of $\overleftrightarrow{BC}$ which contains A, and the side of $\overleftrightarrow{CA}$ which contains B.

Theorem In a Pasch geometry $\text{int}(\triangle ABC)$ is convex.

12. Prove the above theorem.

13. In a Pasch geometry, given $\triangle ABC$ and points D, E, F such that $B - C - D$, $A - E - C$ and $B - E - F$, prove that $F \in \text{int}(\measuredangle ACD)$.

14. In a Pasch geometry, if $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$, prove that either $\overrightarrow{BC} = \overrightarrow{BP}$, or $P \in \text{int}(\measuredangle ABC)$, or $C \in \text{int}(\measuredangle ABP)$.

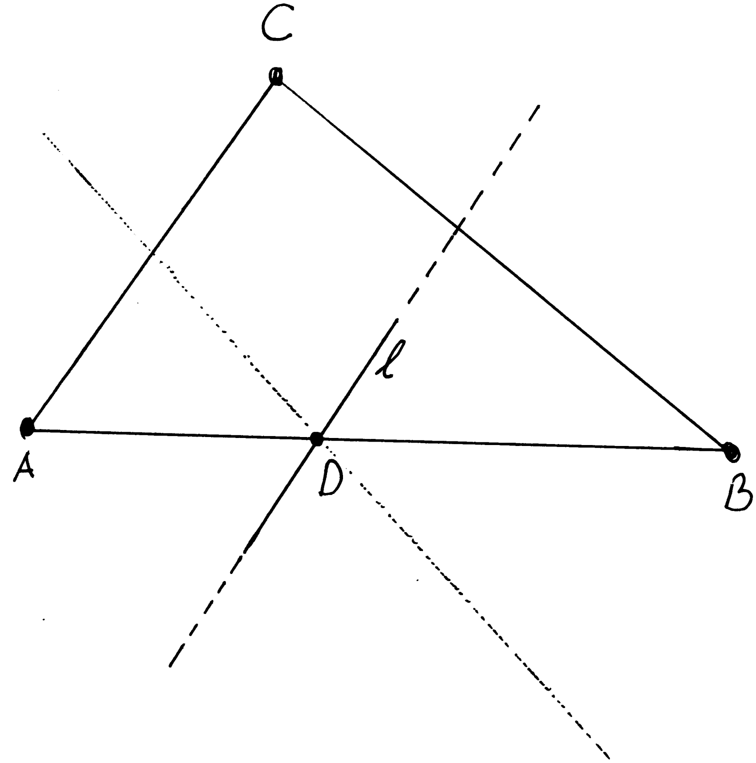
15. Prove that in a Pasch geometry, $\text{int}(\measuredangle ABC)$ is convex.

Pašova geometrija

Definicija (Paš-ov postulat)

Metrična geometrija zadovoljava Pasch-ov postulat (PP) akko za bilo koju pravu ℓ, bilo koji trougao $\triangle ABC$, i bilo koju tačku $D \in \ell$ (takvu da $A-D-B$) imamo

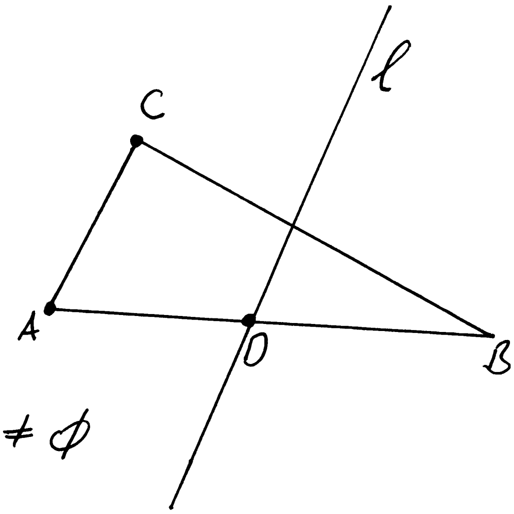
ili $\ell \cap \overline{AC} \neq \emptyset$ ili $\ell \cap \overline{BC} \neq \emptyset$



Teorem (Pasch-ov Teorem)

Ako metrična geometrija zadovoljava PSA tada ona takoter zadovoljava i PP.

Dokazati Paš-ov teorem.



R: Skica dokaza

ΔABC, ℓ

pretp. $\exists D \in \ell$ $A-D-B$

pok. ili $\ell \cap \overline{AC} \neq \emptyset$ ili $\ell \cap \overline{BC} \neq \emptyset$

pretp. $\overline{AC} \cap \ell = \emptyset$. pokaž. $\overline{BC} \cap \ell \neq \emptyset$. $\overline{AC} \cap \ell = \emptyset \Rightarrow A \notin \ell$

$A \in \overline{AC} \cap \overleftrightarrow{AB} \Rightarrow \ell \neq \overleftrightarrow{AB} \Rightarrow A, B \notin \ell$

$$\left.\begin{array}{l} A, B \notin \ell \\ \overline{AB} \cap \ell = \{D\} \neq \emptyset \end{array}\right\} \Rightarrow$$ A i B leže na suprotnim stranama prave ℓ

$\overline{AC} \cap \ell = \emptyset \Rightarrow$ A i C su sa iste strane prave ℓ

$\xRightarrow{\text{prema jednom od ranijih Teorema}}$ B i C su sa različitih strana prave ℓ

$\Rightarrow \overline{BC} \cap \ell \neq \emptyset$.

Prema tome $\overline{AC} \cap \ell \neq \emptyset$ ili $\overline{BC} \cap \ell \neq \emptyset$

Definicija (Pasch-ova geometrija)

Pasch-ova geometrija je metrična geometrija koja zadovoljava PSA.

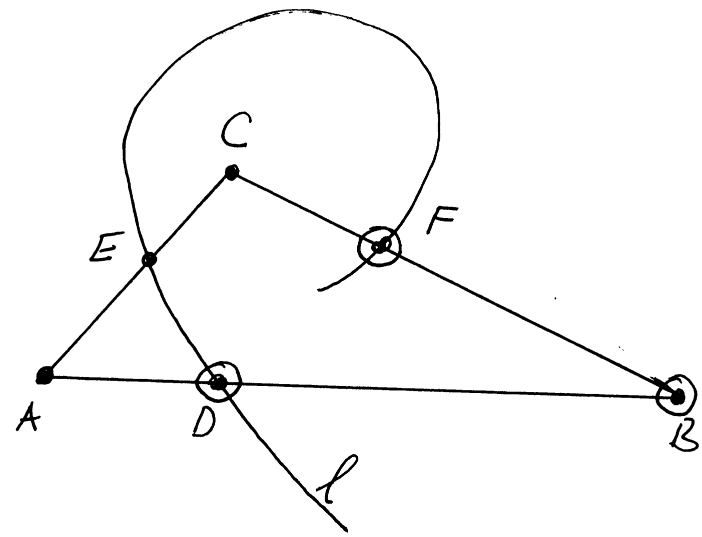
Teorema

Neka je $\{\mathscr{S}, \mathscr{L}, d\}$ metrička geometrija koja zadovoljava PP. Ako su A, B i C nekolinearne i ako prava ℓ ne sadrži ni jednu od tački A, B, C tada prava ℓ ne može sječi sve tri strane trougla ΔABC.

\# Dokazati teoremu iznad.

Rj.:

Skica dokaza



Pretpostavimo suprotno.

$\exists \ell$ t.d.

$\overline{AB} \cap \ell = \{D\}$

$\overline{AC} \cap \ell = \{E\}$

$\overline{BC} \cap \ell = \{F\}$, $A-D-B$, $A-E-C$, $B-F-C$.

$D, E, F \in \ell \Rightarrow$ jedna tačka je između druge dvije

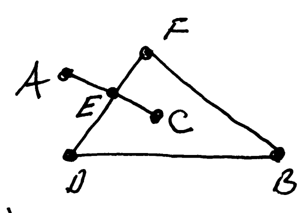
Pretp. $D-E-F$ (slično za ostale slučajeve)

B, D, F nekolin. (u supr. A, B, C kolin.)

ΔBDF

$\overleftrightarrow{AC} \cap \overleftrightarrow{DF} = \{E\}$

$\xRightarrow{\text{Pasch Postulat}}$ $\overleftrightarrow{AC}$ siječe ili $\overline{BD}$ ili $\overline{BF}$



$\overleftrightarrow{AC} \cap \overline{BD} \subseteq \overleftrightarrow{AC} \cap \overleftrightarrow{BA} = \{A\}$

$A \notin \overline{BD}$ (zato što $A-D-B$) $\Rightarrow \overleftrightarrow{AC} \cap \overline{BD} = \emptyset$ ...(1)

S druge strane $\overleftrightarrow{AC} \cap \overline{BF} \subseteq \overleftrightarrow{AC} \cap \overleftrightarrow{BC} = \{C\}$

$B-F-C \Rightarrow C \notin \overline{BF} \Rightarrow \overleftrightarrow{AC} \cap \overline{BF} = \emptyset$ ...(2)

(1) i (2) je u kontradikciji sa PP (primj. ΔBDF, $\overleftrightarrow{AC} \cap \overline{BF} \neq \emptyset$ ili $\overleftrightarrow{AC} \cap \overline{BD} \neq \emptyset$)

Teorema

Ako metrična geometrija zadovoljava PP tada ona također zadovoljava PSA.

(#) Dokazati teoremu iznad.

Rj: Skica dokaza.

ℓ, $P \notin \ell$ (tačka P i prava ℓ postoje zato što u metr. geom $\exists$ tri nekol. tač.)

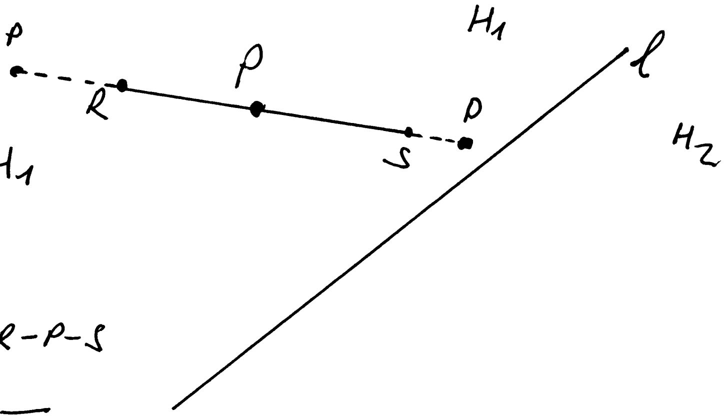
def. H_1, H_2: $H_1 = \{Q \in \mathcal{S} \mid Q = P \text{ ili } \overline{QP} \cap \ell = \emptyset\}$

$H_2 = \{Q \in \mathcal{S} \mid Q \notin \ell \text{ i } \overline{QP} \cap \ell \neq \emptyset\}$

$\Rightarrow H_1 \cap H_2 = \emptyset, \quad \mathcal{S} - \ell = H_1 \cup H_2$

Treb. pok. da su H_1 i H_2 konv. i da zad. (iii) usl. iz def. PSA.

(a) Pok. da je H_1 konv.



$R, S \in H_1$, $R-T-S$. Pok. $T \in H_1$

1° R, S, P kolinearni

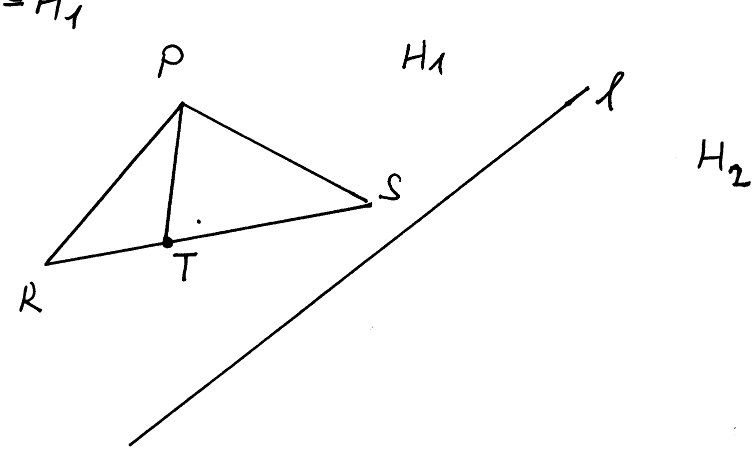
$\Rightarrow R = P,\ S = P,\ R-S-P,\ S-R-P,\ R-P-S$

U svim situacijama $\overline{RS} \subseteq \overline{PR} \cup \overline{PS}$ ...(*)

$R \in H_1 \Rightarrow \overline{PR} \cap \ell = \emptyset \Rightarrow \forall F \in \overline{PR} \quad \overline{PF} \cap \ell = \emptyset \Rightarrow \overline{PR} \subseteq H_1$

Slično $\overline{PS} \subseteq H_1 \overset{(*)}{\Longrightarrow} \overline{RS} \subseteq H_1$

2° R, S i P nisu kolinearni



$\left.\begin{array}{l}\triangle RSP \\ \ell \cap \overline{PS} = \emptyset \text{ i } \ell \cap \overline{PR} = \emptyset\end{array}\right\} \overset{PP}{\Longrightarrow}$

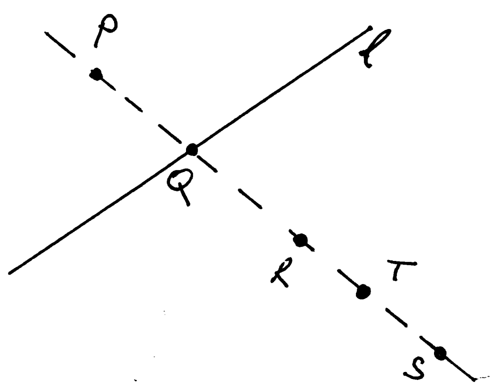
$\overset{PP}{\Longrightarrow} \ell \cap \overline{RS} = \emptyset$

Posm. $\triangle RTP$

$\ell \cap \overline{RS} = \emptyset,\ \overline{RT} \subseteq \overline{RS} \Rightarrow \overline{RT} \cap \ell = \emptyset$

$\overline{RP} \cap \ell = \emptyset \overset{PP\, \triangle RTP}{\Longrightarrow} \overline{PT} \cap \ell = \emptyset \Rightarrow T \in H_1 \Rightarrow H_1$ konveks.

(b) Pok. da je H_2 konv.



$R, S \in H_2 \Rightarrow \overline{PR} \cap l \neq \emptyset, \ \overline{PS} \cap l \neq \emptyset$

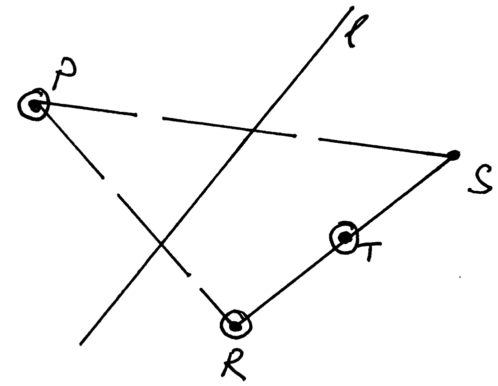
1° R, S, P su kolin. (i različiti)

$\Rightarrow \overline{RP} \cap l = \overline{PS} \cap l = \{Q\}$

i $P-Q-R-S$ ili $P-Q-S-R$

$S-T-R \Rightarrow P-Q-T \Rightarrow \overline{TP} \cap l \neq \emptyset \Rightarrow \overline{RS} \subseteq H_2$

2° R, S, P su nekolinearni



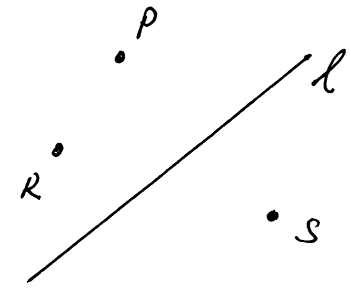
$R-T-S \xrightarrow{\text{pret. Teor.}} T \notin l$ (u suprotnom l siječe sve str. ΔPRS)

Prema istoj Teor. $\Rightarrow l \cap \overline{RS} = \emptyset$

$\Rightarrow \overline{RT} \cap l = \emptyset$

$\left.\begin{array}{l} \Delta PRT \\ l \cap \overline{PR} \neq \emptyset \end{array}\right\} \xRightarrow{PP} \overline{PT} \cap l \neq \emptyset \Rightarrow T \in H_2 \Rightarrow \overline{RS} \subseteq H_2 \Rightarrow H_2$ konv.

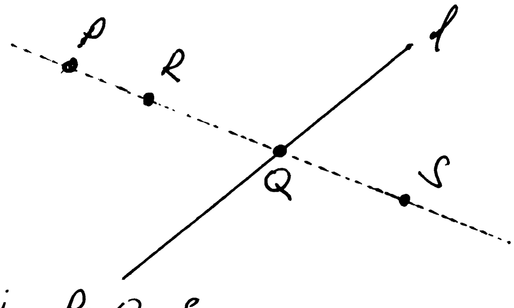
(c) $R \in H_1, S \in H_2$. Pok. da $\overline{RS} \cap l \neq \emptyset$.



$R = P \Rightarrow \overline{RS} \cap l = \overline{PS} \cap l \neq \emptyset$ i zadatak je riješen

Pa pretp. da je $R \neq P$

1° R, S i P nisu kolinearni



$\left.\begin{array}{l} \overline{RP} \cap l = \emptyset \\ \Delta PRS \\ \overline{PS} \cap l \neq \emptyset \end{array}\right\} \xRightarrow{PP} \overline{RS} \cap l \neq \emptyset$

2° R, S, P kolinearni $\xRightarrow{\overline{SP} \cap l \neq \emptyset} \overline{SP} \cap l = \{Q\}$ i $P-Q-S$

$R \in \overleftrightarrow{SP}$ i $R \neq P, R \neq Q, R \neq S \Rightarrow$ ili $P-Q-R$ ili $R-P-Q$ ili $P-R-Q$

$P-Q-R$, $R \in H_1$, $\overline{PR} \cap l = \emptyset$ #kontradikcija $\quad R-P-Q \Rightarrow R-P-Q-S \Rightarrow \overline{RS} \cap l = \{Q\}$

$P-R-Q \Rightarrow P-R-Q-S \Rightarrow \overline{RS} \cap l = \{Q\} \quad \overline{RS} \cap l \neq \emptyset$

(a), (b) i (c) $\Rightarrow$ geom. zad. PSA

(Pean-ov aksiom) Dat je ΔABC u metričkoj geometriji koja zadovoljava PSA i tačke D, E sa osobinama da je $B-C-D$ i $A-E-C$. Pokazati da postoji tačka $F \in \overleftrightarrow{DE}$ takva da $A-F-B$ i $D-E-F$.

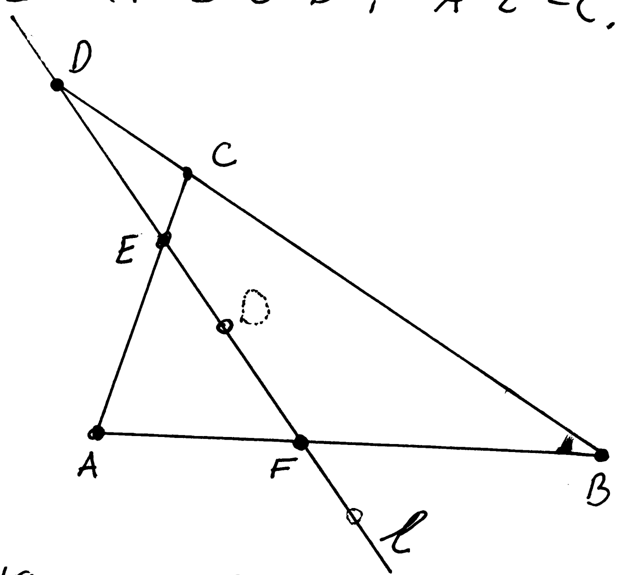
Rj:

I način

Kako data metrička geometrija zadovoljava PSA to ona zadovoljava i PP.

Pa posmatrajmo ΔABC i tačke D, E t.d. $B-C-D$ i $A-E-C$.

Pravu $p(D,E) = \overleftrightarrow{DE}$ označimo sa l



$$\left.\begin{array}{l}\Delta ABC \\ E \in l \\ A-E-C\end{array}\right\} \overset{PP}{\Longrightarrow} \text{ ili } l \cap \overline{AB} \neq \emptyset \quad \text{ili } l \cap \overline{BC} \neq \emptyset$$

S obzirom da prava l siječe pravu $p(B,C) = \overleftrightarrow{BC}$ u tački D i da je $B-C-D$ to prava l ne može sjeći duž $\overline{BC}$. Prema tome $l \cap \overline{AC} \neq \emptyset$ pa neka je F presječna tačka od l i $\overline{AB}$.

Tada $A-F-B$ i $F \in \overleftrightarrow{DE} = l$. Pokažimo još da je $D-E-F$.

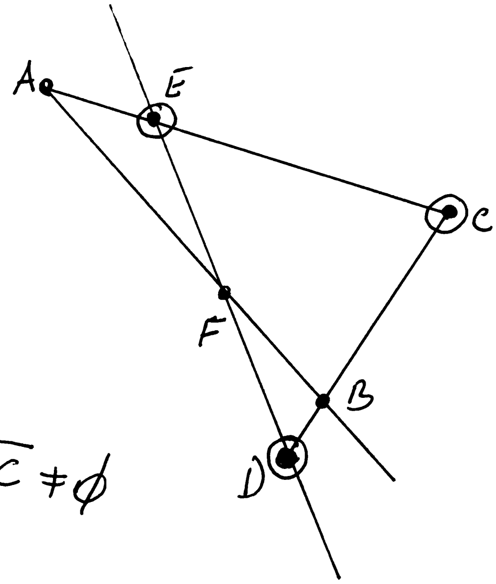
Prvo pokažimo da slučaj $E-D-F$ nije moguć. Ako bi bilo $E-D-F$ imali bi,

$$\left.\begin{array}{l}\Delta AFE \\ D \in p(B,C) \\ E-D-F\end{array}\right\} \overset{PP}{\Longrightarrow} \text{ ili } p(B,C) \cap \overline{AE} \neq \emptyset \quad \text{ili } p(B,C) \cap \overline{AF} \neq \emptyset$$

S obzirom da $p(B,C)$ siječe $p(A,E)$ u tački C i $\overset{\text{vrijedi}}{A-E-C}$ to $p(B,C) \cap \overline{AE} = \emptyset$.
Slično s obzirom da $p(B,C) \cap p(A,F) = \{B\}$ i $A-F-B$ to je $p(B,C) \cap \overline{AF} = \emptyset$.

Ovo je u kontradikciji sa PP pa nije $E-D-F$.

Pokažimo sad da nije $E-F-D$.



Pretpostavimo suprotno, pretpostavimo da jest $E-F-D$. Posmatrajmo $\triangle EDC$

$$\left.\begin{array}{l} \triangle EDC \\ F \in p(A,B) \\ E-F-D \end{array}\right\} \overset{PP}{\Longrightarrow} \quad \begin{array}{l} \text{ili } p(A,B) \cap \overline{CD} \neq \emptyset \\ \text{ili } p(A,B) \cap \overline{EC} \neq \emptyset \end{array}$$

S obzirom da $p(A,B)$ siječe pravu $p(A,C) = \overleftrightarrow{AC}$ u tački A to je $p(A,B) \cap \overline{EC} = \emptyset$. Prema PP to znači da je

$$p(A,B) \cap \overline{CD} \neq \emptyset.$$

Ali kako je $p(A,B) \cap p(C,D) = \{B\}$ to je $p(A,B) \cap \overline{CD} = \{B\}$

$\Rightarrow \quad C-B-D$

\# kontradikcija

(prema pretpostavci zadatka imamo da je $B-C-D$).

Prema tome nije $E-F-D$.

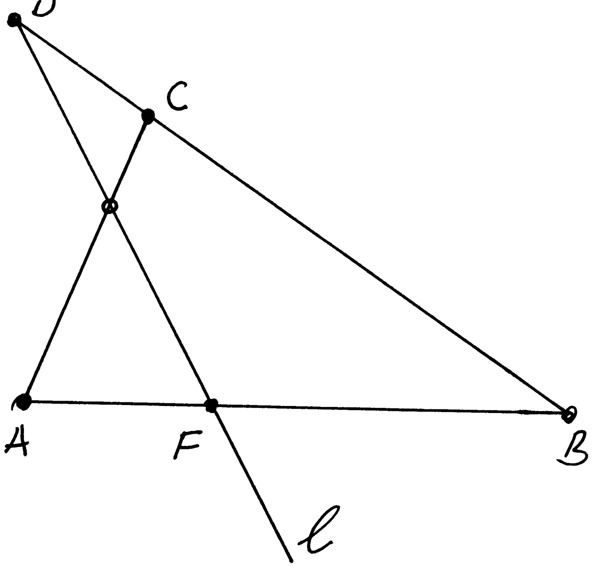
Kako nije $E-D-F$ i nije $E-F-D$ to mora biti $D-E-F$

š.e.d.

II način

Pokušati zadatak riješiti bez upotrebe PP, samo uz upotrebu PSA.

(#) Dat je ΔABC u metričnoj geometriji koja zadovoljava PSA i date su tačke D, F sa osobinom da je $B-C-D$ i $A-F-B$. Pokazati da postoji tačka $E \in \overleftrightarrow{DF}$ sa osobinom da $A-E-C$ i $D-E-F$.



Rj:

S obzirom da data metrična geometrija zadovoljava PSA to ona zadovoljava i PP.

Pravu $\ell(D,F) = \overleftrightarrow{DF}$ označimo sa ℓ i posmatramo ΔABC.

$$\left.\begin{array}{c} \Delta ABC \\ F \in \ell \\ A-F-B \end{array}\right\} \overset{PP}{\Longrightarrow} \text{ ili } \overline{AC} \cap \ell \neq \phi \text{ ili } \overline{BC} \cap \ell \neq \phi. \qquad \ldots (*)$$

S obzirom da ℓ siječe pravu $\ell(B,C) = \overleftrightarrow{BC}$ u tački D i vrijedi $B-C-D$ to $\ell \cap \overline{BC} = \phi$. Na osnovu $(*)$ to znači da je $\ell \cap AC \neq \phi$, pa neka je $\ell \cap AC = \{E\}$. Time imamo da je

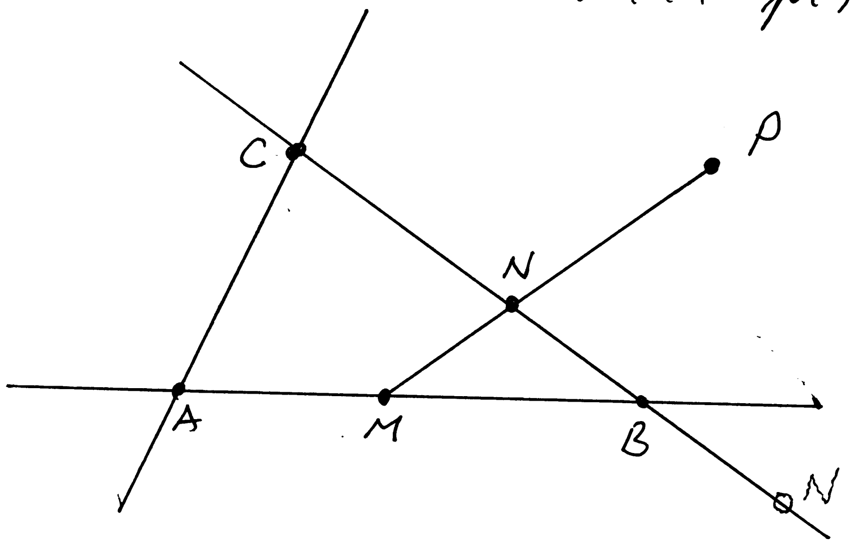
$$E \in \overleftrightarrow{DF} \text{ i } A-E-C.$$

Dokaz da je $D-E-F$ je potpuno isti kao u prethodnom zadatku (pretpostavimo da je $E-D-F$, posmatramo ΔAFE i $\ell(B,C)$ - dobijemo kontradikciju; pretpostavimo da je $E-F-D$, posmatramo ΔEDC i pravu $\ell(A,B)$ i dobijemo kontradikciju).

(#) Dat je ΔABC i data je tačka P u metričnoj geometriji koja zadovoljava PSA. Dokazati da postoji prava kroz tačku P koja sadrži tačno dvije tačke ΔABC.

Rj. S obzirom da ne znamo "položaj" tačke P, to zadatak možemo podjeliti u više slučajeva. Ovdje ćemo posmatrati samo jedan slučaj, a svi ostali slučajevi se izvode analogno.

Pa pretpostavimo da $P \notin \ell(A,B) = \overleftrightarrow{AB}$, $P \notin \ell(A,C) = \overleftrightarrow{AC}$, $P \notin \ell(B,C) = \overleftrightarrow{BC}$, da se tačka P i A nalaze sa različitih strana $\ell(B,C) = \overleftrightarrow{BC}$, da se tačka B i P nalazi sa iste strane prave $\ell(A,C) = \overleftrightarrow{AC}$ i da se tačke P i C nalaze sa iste strane prave $\ell(A,B) = \overleftrightarrow{AB}$.



Neka je $M \in \ell(A,B)$ t.d. $A-M-B$.

Kako data metrička geometrija zadovoljava PSA to ona zadovoljava i PP. Posmatrajmo ΔABC i $\ell(P,M)$.

$$\left.\begin{array}{l} \Delta ABC \\ M \in \ell(M,P) \\ A-M-B \end{array}\right\} \overset{PP}{\Rightarrow} \text{ ili } \ell(M,P) \cap \overline{AC} \neq \emptyset \quad \text{ ili } \ell(M,P) \cap \overline{BC} \neq \emptyset.$$

S obzirom da je $B-M-A$ i $M \notin \ell(B,C)$ to su M i A sa iste strane $\ell(B,C)$. $\Rightarrow$ P i M sa različitih strana $\ell(B,C)$.

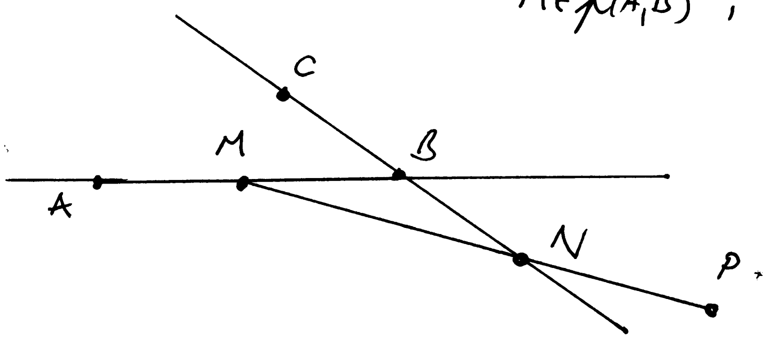
$\Rightarrow \quad \overline{PM} \cap p(B,C) \neq \emptyset$

Pa njihov presjek označimo sa N.

S obzirom da $P \notin p(A,B)$ i $P \in p(A,C)$ to je $N \neq B$ i $N \neq C$.
To znači da su mogući slučajevi 1° $C-N-B$
2° $C-B-N$
3° $N-C-B$

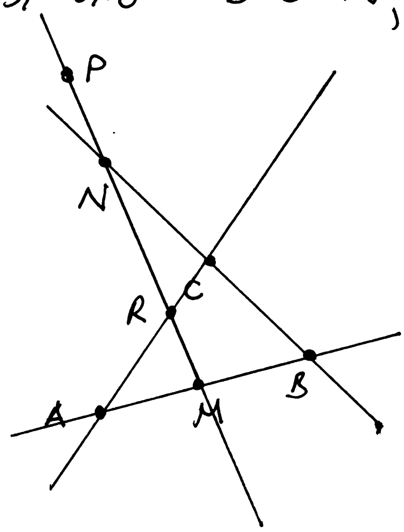
Ako bi bilo $C-B-N$ $\Rightarrow$ C i N su sa različitih strana prave $p(\,,B)$. ...(1)

$M \in p(A,B)$ i $M-N-P$ $\Rightarrow$ N i P su sa iste strane prave $p(\,,B)$ ...(2)



(1) i (2) $\Rightarrow$ P i C su sa različitih strana prave $p(A,B)$
#kontradikcija.

Slično, ako bi bilo $B-C-N$, s obzirom da je $A-M-B$, prema prethodnom zadatku imamo da $\exists R$ t.d. $A-R-C$ i $M-R-N$.



$$\left.\begin{array}{l} M-R-N \\ M-N-P \end{array}\right\} \Rightarrow M-R-N-P.$$

$B-C-N$ $\Rightarrow$ B i N su sa različitih strana $p(A,C)$ ...(3)

$R \in p(A,C)$ i $R-N-P$ $\Rightarrow$ N i P su sa iste strane $p(A,C)$ ...(4)

(3) i (4) $\Rightarrow$ P i B su sa različitih strana $p(A,C)$
#kontradikcija.

Prema tome mora vrijediti $B-N-C$, i prava $p(P,M) = \overleftrightarrow{PM}$ siječe ΔABC u tačkama M i N.

Definicija (missing strip ravan)

Missing strip ravan je apstraktna geometrija $\{\mathscr{S}, \mathscr{L}\}$ data sa

$$\mathscr{S} = \{(x,y) \in \mathbb{R}^2 \mid x<0 \text{ ili } 1 \leq x\},$$

$$\mathscr{L} = \{\ell \cap \mathscr{S} \mid \ell \text{ je Dekartova ravan i } \ell \cap \mathscr{S} \neq \emptyset\}.$$

Dati su sljedeći parovi tački:

(i) $(2,3)$ i $(3,-1)$;

(ii) $(0,3)$ i $\left(\frac{1}{2},-2\right)$;

(iii) $(-1,4)$ i $(2,7)$

Ako dati par tački leži u skupu tački Missing strip ravni, pronaći pravu koja sadrži taj par tački.

Rj.

Missing strip ravan za skup tački ima skup

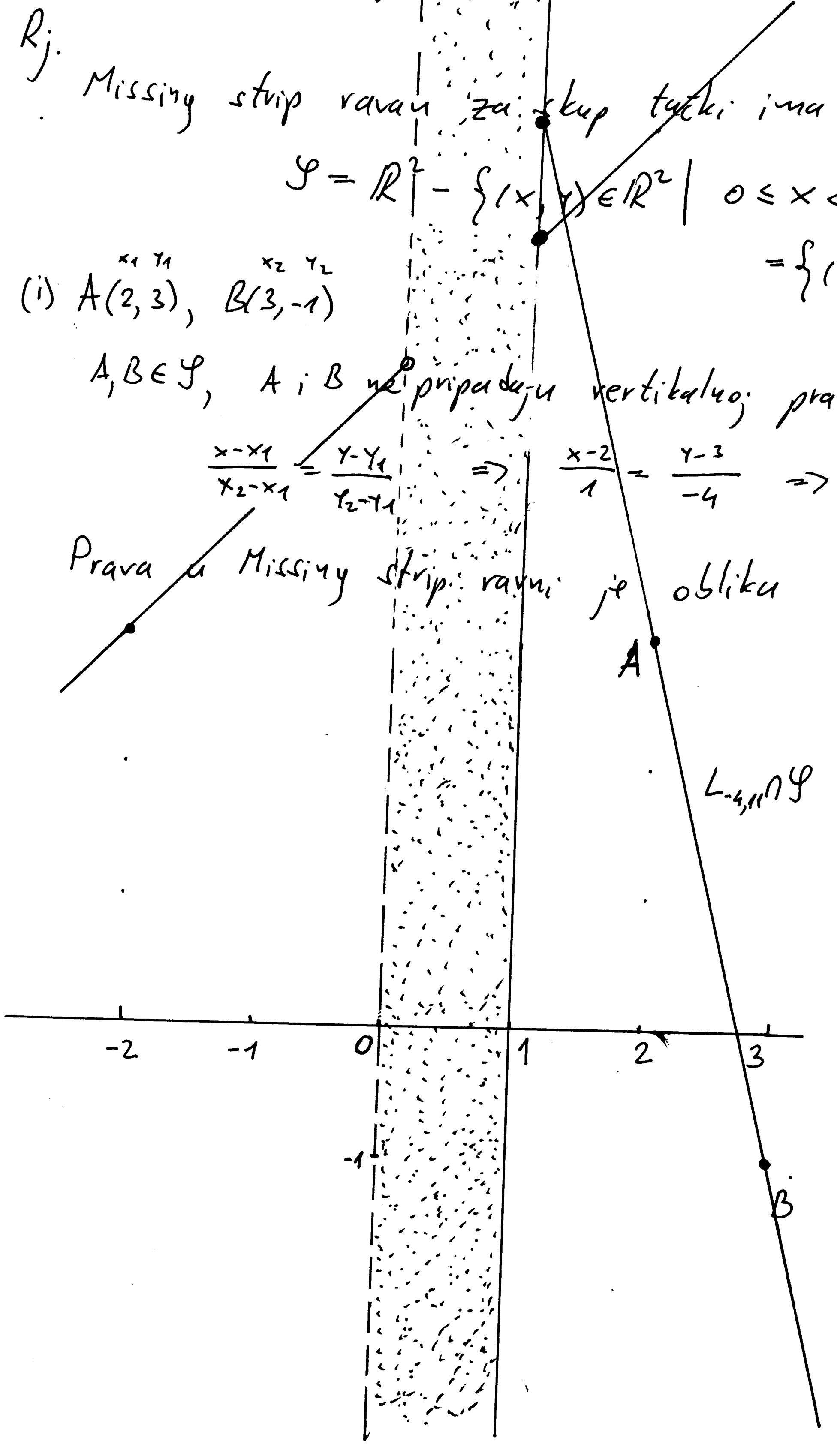
$$\mathcal{S} = \mathbb{R}^2 - \{(x,y)\in\mathbb{R}^2 \mid 0 \le x < 1\} =$$
$$= \{(x,y)\in\mathbb{R}^2 \mid x<0 \text{ i } 1 \le x\}$$

(i) $A(\overset{x_1}{2},\overset{y_1}{3})$, $B(\overset{x_2}{3},\overset{y_2}{-1})$

$A,B\in\mathcal{S}$, A i B ne pripadaju vertikalnoj pravoj

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \Rightarrow \frac{x-2}{1} = \frac{y-3}{-4} \Rightarrow y=-4x+11 \text{ Euklidova prava}$$

Prava u Missing strip ravni je oblika $L_{-4,11}\cap\mathcal{S}$



(ii) $M(0,3)$, $N\left(\frac{1}{2},-2\right)$

$M\notin\mathcal{S}$, $N\in\mathcal{S}$

(iii) $P(-1,4)$, $R(2,7)$

Euklidova prava je $L_{1,5}$

$y=x+5$

Prava u Missing strip ravni je

$L_{1,5}\cap\mathcal{S}$

($\mathcal{S}$ = skup tački Missing strip ravni)

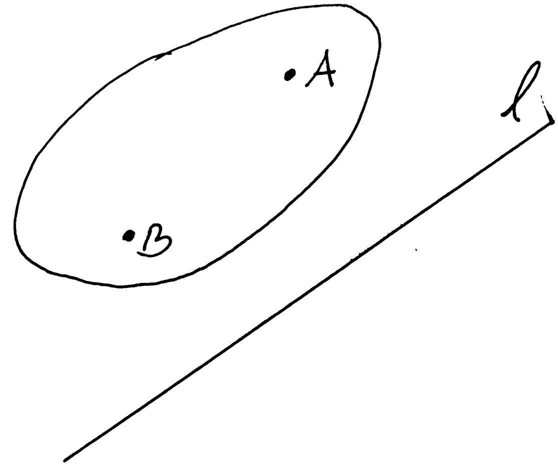
Unutrašnjosti i crossbar teorem

Teorema

U Pasch-ovoj geometriji, ako je $\mathcal{A}$ neprazan konveksan skup koji ne siječe pravu l, tada sve tačke od $\mathcal{A}$ leže na istoj strani od l.

\# Dokazati teoremu iznad.

Rj.



Skica dokaza.

$A \in \mathcal{A}$, B bilo koja druga tačka $\in \mathcal{A}$

$\mathcal{A}$ konv. $\Rightarrow \overline{AB} \subseteq \mathcal{A}$

$\mathcal{A} \cap l = \emptyset \Rightarrow \overline{AB} \cap l = \emptyset \Rightarrow$ A i B su sa iste strane prave l

$\Downarrow$

svaka tačka je sa iste strane prave l sa koje je i tačka A

Definicija (unutrašnjost poluprave, unutrašnjost duži)

Unutrašnjost poluprave $pp[A, B) = \overrightarrow{AB}$ u metričkoj geometriji je skup

$$int(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}.$$

Unutrašnjost duži $\overline{AB}$ u metričnoj geometriji je skup

$$int(\overline{AB}) = \overline{AB} - \{A, B\}.$$

#) Pokaži da su u metričnoj geometriji $int(\overrightarrow{AB})$ i $int(\overline{AB})$ konveksni skupovi.

Rj.

$$pp[A,B) = \overrightarrow{AB} = \overline{AB} \cup \{C \in \mathcal{S} \mid A-B-C\}$$

$$\overline{AB} = \{M \in \mathcal{S} \mid A-M-B \text{ ili } M=A \text{ ili } M=B\}$$

$$\begin{aligned} int(\overrightarrow{AB}) &= \overrightarrow{AB} - \{A\} \\ &= \{M \in \mathcal{S} \mid A-M-B \text{ ili } M=B \text{ ili } A-B-M\} \end{aligned}$$

Izaberimo proizvoljne dvije tačke $P, Q \in int(\overrightarrow{AB})$, $P \neq Q$.

$$P \in int(\overrightarrow{AB}) \Rightarrow A-P-B \text{ ili } P=B \text{ ili } A-B-P.$$

$$Q \in int(\overrightarrow{AB}) \Rightarrow A-Q-B \text{ ili } Q=B \text{ ili } A-B-Q.$$

Mogući slučajevi su:

1° $A-P-B$; $A-Q-B$
2° $A-P-B$; $Q=B$
3° $A-P-B$; $A-B-Q$
4° $P=B$; $A-Q-B$
5° $P=B$; $A-B-Q$
6° $A-B-P$; $A-Q-B$
7° $A-B-P$; $Q=B$
8° $A-B-P$; $A-B-Q$.

Riješimo npr. prvi slučaj. Svi ostali se rješavaju na isti način

$$\left.\begin{matrix} A-P-B \\ A-Q-B \end{matrix}\right\} \Rightarrow \forall T \text{ (za koje je } P-T-Q) \text{ imamo da je } A-T-B$$

$$\Downarrow$$

$$T \in int(\overrightarrow{AB})$$

$$\Downarrow$$

$$\overline{PQ} \subseteq int(\overrightarrow{AB})$$

$$\Downarrow$$

$int(\overrightarrow{AB})$ konv. skup

Slično za $int(\overline{AB})$.

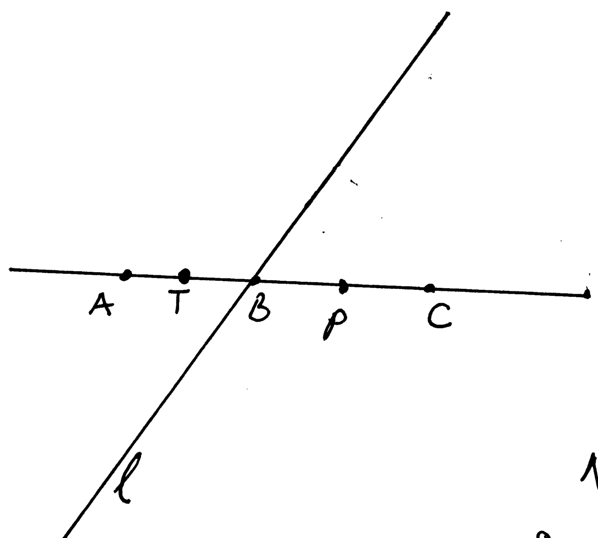
$$int(\overline{AB}) = \{C \in \mathcal{S} \mid A-C-B\}.$$

Teorema

Neka je A prava, poluprava, duž, unutrašnjost poluprave ili unutrašnjost duži u Pasch-ovoj geometriji. Ako je l prava takva da $A \cap l = \emptyset$ tada cijeli A leži na istoj strani prave l. Ako postoji tačka B takva da $A-B-C$ i $\overleftrightarrow{AC} \cap l = \{B\}$ tada int($\overrightarrow{BA}$) i int($\overline{BA}$) oba pripadaju istoj strani prave l dok int($\overrightarrow{BA}$) i int($\overrightarrow{BC}$) pripadaju suprotnim stranama prave l.

(#) Dokazati teoremu iznad.

Rj. Bez obzira da li je A prava ili poluprava ili... skup A je konveksan skup, pa ako je $A \cap l = \emptyset$ prema prethodnoj teoremi cijeli A leži sa iste strane prave l.



Neka je $T \in \overleftrightarrow{AC}$ t.d. $A-T-B$.
Ako je $\overleftrightarrow{AC} \cap l = \{B\}$ prema istoj teoremi int($\overrightarrow{BA}$) i int($\overline{BA}$) se nalaze sa one strane prave l sa koje je i tačka T.

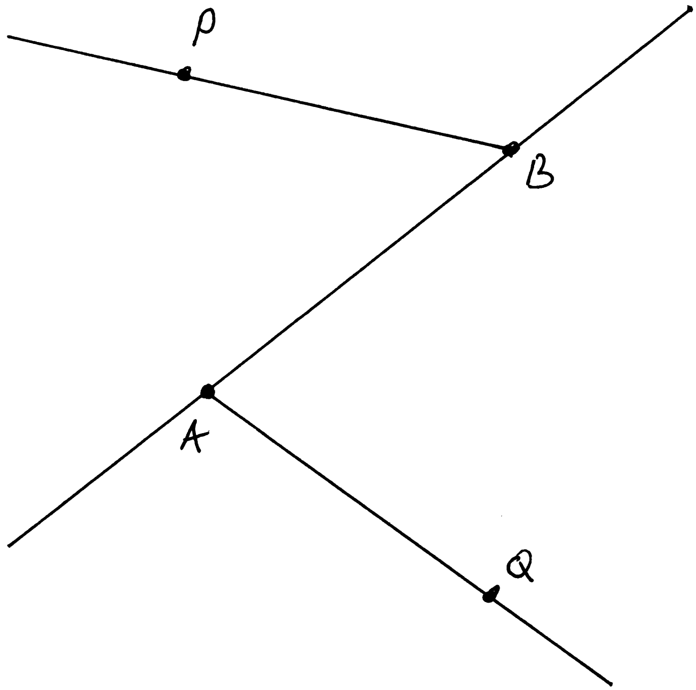
Neka je $P \in \overleftrightarrow{AC}$ t.d. $B-P-C$.
S obzirom da je $B \in l$ i $T-B-P$ to su T i P sa različitih strana prave l.
Sad nije teško vidjeti da prema istoj teoremi int($\overrightarrow{BA}$) pripada onoj strani prave l sa koje je i tačka T, dok int($\overrightarrow{BC}$) pripada onoj strani prave l sa koje je tačka P. Slijedi da int($\overrightarrow{BA}$) i int($\overrightarrow{BC}$) pripadaju suprotnim stranam prave l.

Teorem (Z Teorem)

U Pašovoj geometriji, ako su P i Q sa različitih strana prave $p(A,B) = \overleftrightarrow{AB}$ tada

$$\overrightarrow{BP} \cap \overrightarrow{AQ} = \emptyset.$$

U stvari, $\overline{BP} \cap \overline{AQ} = \emptyset$.



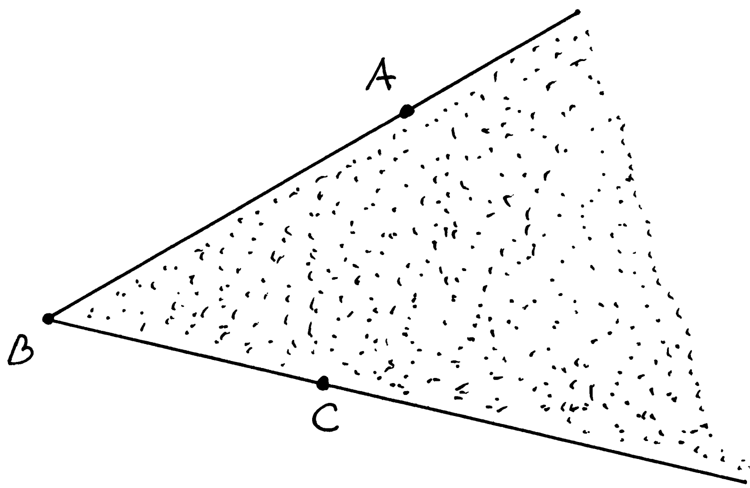
(#) Dokazati teoremu iznad.

Rj:

Dokaz pronađi u knjizi Teorem 4.4.3.

<u>Definicija</u> (unutrašnjost $\sphericalangle ABC$)

U Pasch-ovoj geometriji unutrašnjost $\sphericalangle ABC$ (što označavamo sa $\mathrm{int}(\sphericalangle ABC)$) je presjek strane prave $p(A,B) = \overleftrightarrow{AB}$ koja sadrži tačku C sa stranom prave $\overleftrightarrow{BC}$ koja sadrži tačku A.



Teorema

U Pasch-ovoj geometriji, ako je $\measuredangle ABC = \measuredangle A'B'C'$ tada je $\operatorname{int}(\measuredangle ABC) = \operatorname{int}(\measuredangle A'B'C')$.

\# Dokaži teoremu iznad.

Rj: Prisjetimo se

· Teorema U metričnoj geometriji, ako je $\measuredangle ABC = \measuredangle DEF$ tada je $B = E$.

Teorem U metričkoj geometriji
(i) Ako je $C \in \overrightarrow{AB}$ i $C \neq A$ tada $\overrightarrow{AC} = \overrightarrow{AB}$
(ii) Ako je $\overrightarrow{AB} = \overrightarrow{CD}$ tada je $A = C$.

Ostatak dokaza vidi u knjiti (Teorema 4.4.4.).

Teorema

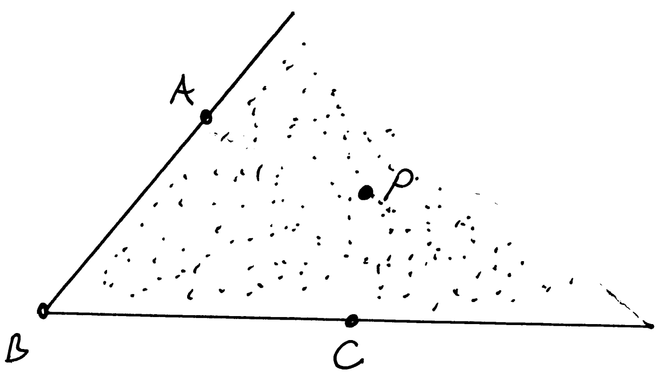
U Pasch-ovoj geometriji $P \in int(\measuredangle ABC)$ ako i samo ako su A i P sa iste strane prave $\overleftrightarrow{BC}$ i C i P su sa iste strane prave $\mu(B,A) = \overleftrightarrow{BA}$.

(#) Dokazati teoremu iznad.

Rj.

„$\Rightarrow$" Pretpostavimo da $P \in int(\measuredangle ABC)$.

Prema definiciji: $int(\measuredangle ABC) = \{$sve tačke sa one strane prave $\mu(A,B) = \overleftrightarrow{AB}$ sa koje je i tačka $C\} \cap \{$sve tačke sa one strane prave $\mu(B,C) = \overleftrightarrow{BC}$ sa koje je i tačka $A\}$



$P \in int(\measuredangle ABC)$

$\Downarrow$

$P \in \{$onim tačkama sa one strane prave $\overleftrightarrow{AB}$ sa koje je i tačka $C\}$ i $P \in \{$onim tačkama sa one strane prave $\overleftrightarrow{BC}$ sa koje je i tačka $A\}$

$\Rightarrow \overline{PC} \cap \overleftrightarrow{AB} = \emptyset$ i $\overline{PA} \cap \overleftrightarrow{BC} = \emptyset \Rightarrow$ A i P su sa iste strane prave $\overleftrightarrow{BC}$ i C i P su sa iste strane prave $\mu(B,A) = \overleftrightarrow{BA}$.

„$\Leftarrow$" Pretpostavimo da su A i P sa iste strane prave $\overleftrightarrow{BC}$, i da su C i P sa iste strane prave $\overleftrightarrow{BA}$.

$\Rightarrow \overline{AP} \cap \overleftrightarrow{BC} = \emptyset$ i $\overline{CP} \cap \overleftrightarrow{BA} = \emptyset$

$\Downarrow$ $P \in$ onoj strani prave $\overleftrightarrow{BC}$ sa koje je tačka A

$\Downarrow$ $P \in$ onoj strani prave $\overleftrightarrow{BA}$ sa koje je tačka C

$\Rightarrow C \in int(\measuredangle ABC)$.

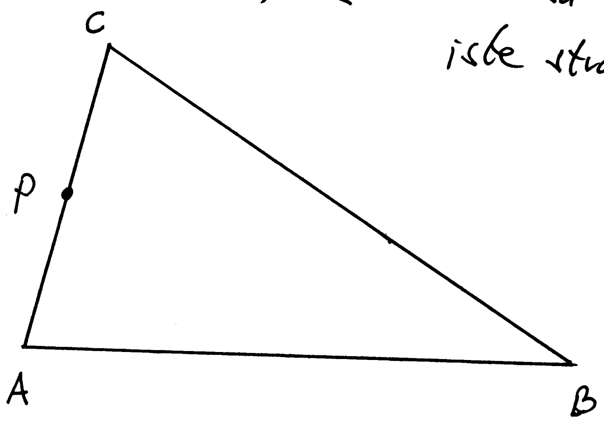
Teorema

U Pasch-ovoj geometriji dat je $\triangle ABC$. Ako je $A-P-C$ tada je $P \in int(\sphericalangle ABC)$ a time $int(\overline{AC}) \subseteq int(\sphericalangle ABC)$.

Dokazati teoremu iznad

Rj. U rješenju zadatka ćemo upotrebiti prethodnu teoremu

· $P \in int(\sphericalangle ABC) \iff$ A i P su sa iste strane prave $\overleftrightarrow{BC}$; a C i P su sa iste strane prave $\overleftrightarrow{BA}$.



$(\overline{CP} \cap \overleftrightarrow{AB} = \phi)$

$A-P-C \overset{A \in \overleftrightarrow{AB}}{\Longrightarrow}$ tačke P i C pripadaju istoj strani prave $\overleftrightarrow{AB}$ ...(1)

$C-P-A \Rightarrow \overline{AP} \cap \overleftrightarrow{BC} = \phi \overset{C \in \overleftrightarrow{BC}}{\Longrightarrow}$ tačke A i P pripadaju istoj strani prave $\overleftrightarrow{BC}$ ...(2)

(1) i (2) $\overset{Teor.}{\Longrightarrow} P \in int(\sphericalangle ABC)$

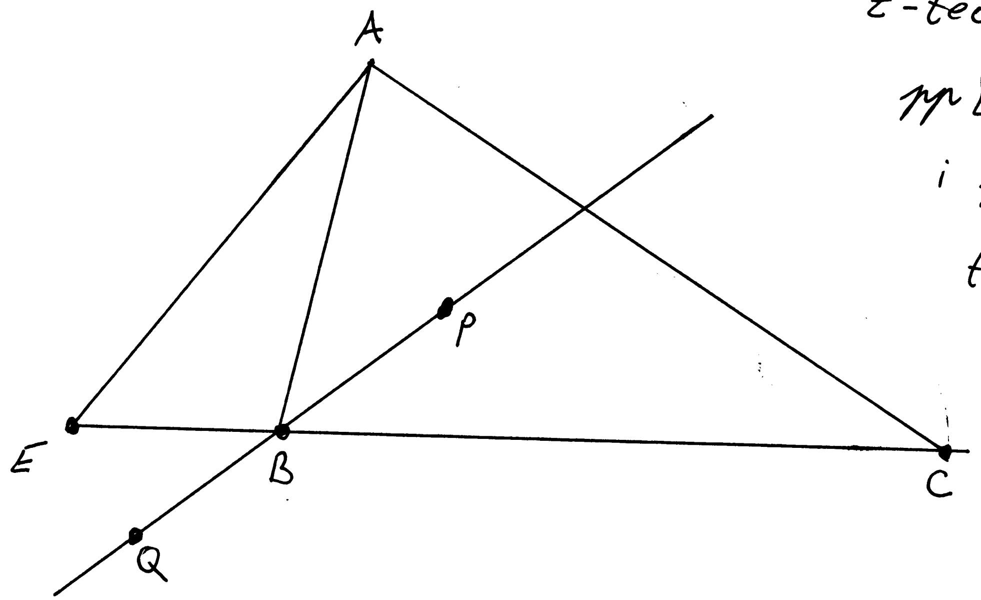
Kako je P proizvoljna tačka i $P \in int(\overline{AC}) \Rightarrow int(\overline{AC}) \subseteq int(\sphericalangle ABC)$.

Teorem (crossbar teorem)

U Pašovoj geometriji ako je $P \in \mathrm{int}(\measuredangle ABC)$ tada $\overrightarrow{BP}$ prava siječe duž $\overline{AC}$ u jedinstvenoj tački F sa osobinom $A-F-C$.

\# Pokazati teoremu iznad.

Rj: Dokaz se nalazi u knjizi (Teorem 4.4.7.)



Dokaz se svodi na to da dva puta upotrebimo Z-teorem (na $pp[A,E)$ i $pp[B,P)$, kao i na $pp[E,A)$ i $pp[B,Q)$) a poslije toga da primjenimo Pašov postulat na ΔECA.

Teorema

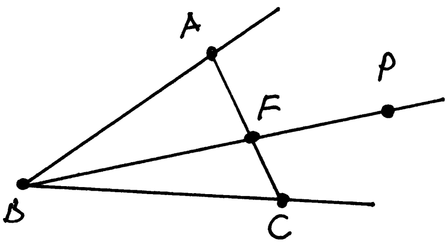
U Pasch-ovoj geometriji, ako je $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$ tada $P \in int(\sphericalangle ABC)$ ako i samo ako su A i C sa različitih strana prave $\overleftrightarrow{BP}$.

Dokazati teoremu iznad.

Rj.

„$\Rightarrow$" Neka je $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$ i pretpostavimo da $P \in int(\sphericalangle ABC)$

$$P \in int(\sphericalangle ABC) \xRightarrow{\text{Crossbar teor.}} \overrightarrow{BP} \cap \overline{AC} = \{F\}, \ A-F-C$$

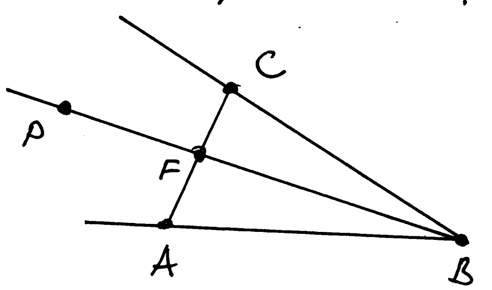


$$\Downarrow F \in p(B,P) = \overleftrightarrow{BP}$$

A i C su sa različitih strana prave $\overleftrightarrow{BP}$, q.e.d.

$(\overline{AC} \cap \overleftrightarrow{BP} \neq \emptyset)$

„$\Leftarrow$" Neka je $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$ i pretpostavimo da su A i C sa različitih strana prave $\overleftrightarrow{BP}$.



$\overline{CP} \cap \overleftrightarrow{AB} = \emptyset \Rightarrow$ tačke C i P su sa iste strane prave $\overleftrightarrow{AB}$ ...(1)

A i C sa razl. str. pr. $\overleftrightarrow{BP} \Rightarrow \overline{AC} \cap \overleftrightarrow{BP} \neq \emptyset \Rightarrow$

$\Rightarrow \overline{AC} \cap \overleftrightarrow{BP} = \{F\}$ i s obzirom da je $\overline{PC} \cap \overleftrightarrow{AB} = \emptyset$ to je moguć tačno jedan od sljedeća tri slučaja

1° $P-F-B$

2° $P=F$

3° $F-P-B$

Posmatrajmo prvi slučaj $P-F-B$ i pokažimo da su tačke P, A sa iste strane prave $\overleftrightarrow{BC}$.

Posmatrajmo pravu $\overleftrightarrow{BP}$. Kako su A i C sa suprotnih strana ove prave to je prema Z teoremu $\overline{PA} \cap \overrightarrow{BC} = \emptyset$. $\quad \ldots(*)$

Posmatrajmo pravu $\overleftrightarrow{AC}$. Kako su P i B sa suprotnih strana ove prave to je $\overline{PA} \cap \overrightarrow{CB} = \emptyset$ $\quad \ldots(**)$

$$(*) \text{ i } (**) \Rightarrow \overline{PA} \cap \overleftrightarrow{BC} = \emptyset$$

$$\Downarrow$$

tačke P i A su sa iste strane prave BC $\quad \ldots(2)$

$$(1) \text{ i } (2) \Rightarrow P \in \operatorname{int}(\measuredangle ABC)$$

Slučajeve 2° i 3° ostavljamo za vježbu.

5. Given the following pairs of points: (i) $(2,3)$ and $(3,-1)$; (ii) $(0,3)$ and $(\frac{1}{2},-2)$; (iii) $(-1,4)$ and $(2,7)$.
For both the Moulton Plane and the Missing Strip Plane, *if* the given pair of points lies in the point set for that geometry, find the line through that pair of points.

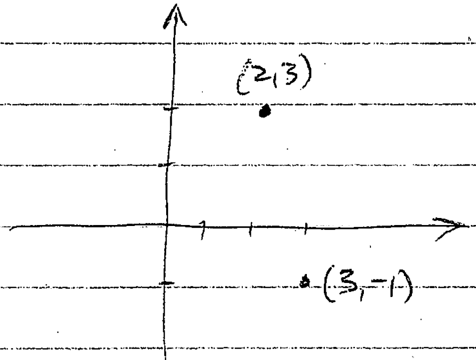
---

(5) (i) $(2,3)$ and $(3,-1)$.

Moulton Plane:

negative slope, so line is same as Euclidean line:

Slope is $\dfrac{3-(-1)}{2-3} = -4$



So $y=-4x+k$ where (through $(2,3)$): $3=-4\cdot 2+k$, $k=11$

so $y=-4x+11$, or $L_{-4,11}$.

Missing Strip plane: same: points are in plane, so line is $L_{-4,11}\cap\mathcal{P}$ where $\mathcal{P}$ = pts in missing strip plane.

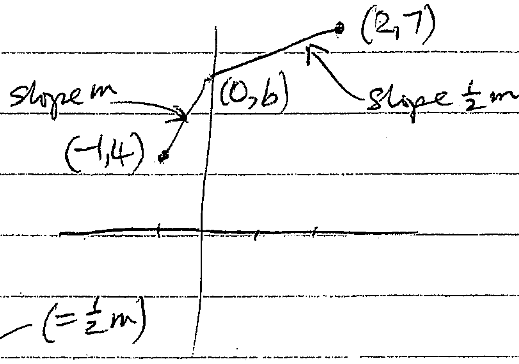
(ii) $(0,3)$ and $(\frac{1}{2},-2)$. Latter pt isn't in the Missing Strip plane

For Moulton Plane:

Slope is $\dfrac{3-(-2)}{0-\frac{1}{2}} = \dfrac{5}{-\frac{1}{2}} = -10$, negative, so line is same as Euclidean line; $y=-10x+3$ or $L_{-10,3}$.

5(iii) $(-1, 4)$ and $(2, 7)$.

Slope is $\dfrac{7-4}{2-(-1)} = \dfrac{3}{3} = 1$, for Euclidean Line.



Moulton plane

Take $(0,b)$ so that $(-1,4)$ to $(0,b)$ has slope

$$\frac{b-4}{0-(-1)} = b-4 \quad (=m)$$

and $(0,b)$ to $(2,7)$ has slope $\left[\dfrac{7-b}{2}\right]$ $(=\tfrac{1}{2}m)$, & we require

$$b-4 = 2\left(\frac{7-b}{2}\right) \quad \text{or} \quad b-4 = 7-b, \text{ so } \underline{b = 11/2}$$

Then the Moulton line is $M_{3/2,\, 11/2}$

slope $b-4$; intercept.

Missing Strip plane

Both points are in this plane.

The line is the Euclidean line intersected with the point set,

so $y = x + k$ where (use $(-1,4)$): $4 = -1 + k$, $k = 5$

so the line is $L_{1,5} \cap \mathcal{P}$

where $\mathcal{P}$ is the point set for the Missing Strip plane.

B1 *(4 marks)* If lines ℓ_1, ℓ_2 and ℓ_3 in the Missing Strip plane, where the *missing* strip is $\{(x,y) \mid 0 \leqslant x < 1\}$, satisfy:

$$\ell_1 \text{ is parallel to } \ell_2 \text{ and } \ell_2 \text{ is parallel to } \ell_3,$$

is it true that ℓ_1 is parallel to ℓ_3? Justify your answer.

False. One example suffices.

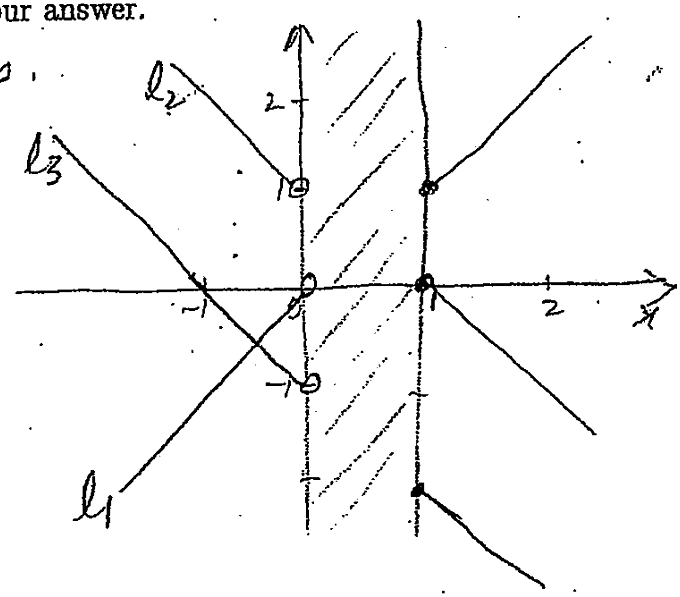
Let $\ell_1 = L_{1,0} \cap S$,

$\ell_2 = L_{-1,1} \cap S$,

$\ell_3 = L_{-1,-1} \cap S$.

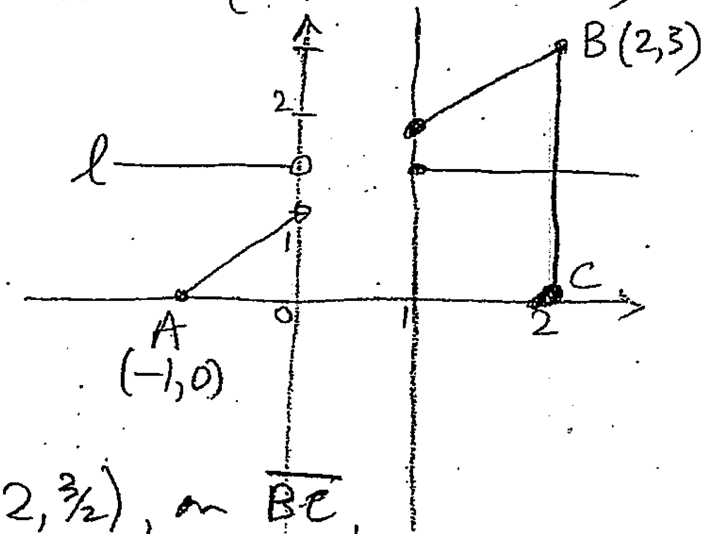
Then $\ell_1 \parallel \ell_2$, $\ell_2 \parallel \ell_3$,

but $\ell_1 \nparallel \ell_3$.



(b) *(3 marks)* <u>Given</u> that a metric geometry satisfies PSA if and only if it is a Pasch geometry, give an example to show that the Missing Strip Plane does not satisfy PSA. (Recall that the point set is $\mathbb{R}^2 \setminus \{(x, y) \mid 0 \leqslant x < 1\}$.)

Enough to take $\triangle ABC$, and line ℓ which hits one side of $\triangle ABC$ (not at a vertex) yet no other side



Eg. $A = (-1, 0)$
$B = (2, 3)$
$C = (2, 0)$

$\ell = L_{0,3/2} \cap S.$

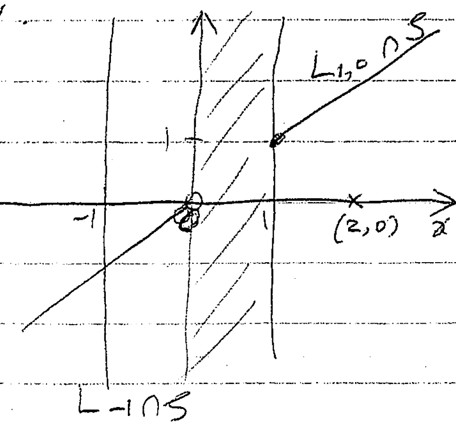
ℓ meets $\triangle ABC$ <u>only</u> at $(2, 3/2)$, on $\overline{BC}$,
but not on $\overline{AC}$ and not on $\overline{AB}$.

7. If $\mathcal{S}$ denotes the set of points of the missing strip plane, find the lines through the point $(2,0)$ which are parallel in the missing strip plane to:

(a) the line $L_{-1} \cap \mathcal{S}$; (b) the line $L_{1,0} \cap \mathcal{S}$.

(7) Missing Strip Plane. Want lines through $(2,0)$, parallel to (a) $L_{-1} \cap S$, (b) $L_{1,0} \cap S$.

(a) Recall $S = \mathbb{R}^2 \setminus \{(x,y) \mid 0 \le x < 1\}$.



Only the line $L_2 \cap S$ $(= L_2)$ is parallel to $L_{-1} \cap S$ & goes through $(2,0)$.

(b) See picture! We'll get more "parallel" lines here.

7(b). Certainly $L_{1,-2} \cap S$ is parallel to $L_{1,0} \cap S$, and it goes through $(2,0)$. (Same as in Euclidean plane: $L_{1,-2}$ and $L_{1,0}$ are parallel — have same slope 1.)

But there are more!

Consider lines with 0 or negative slope, $L_{-m,b}$, where $y = -mx + b$, & make line go through $(2,0)$.

So $0 = -2m + b$ or $b = 2m$.

So lines $L_{-m,2m}$ (or $y = -mx + 2m$) have the potential to be parallel. What can m be?

(Looking at the figure to see the "gap" in the missing strip will help here!)

When $x = 0$ (NOT in the plane — part of the "missing" strip) we have $y = 2m$.

And when $x = 1$ (just IN the plane), $y = -m + 2m = m$.

So to avoid $L_{1,0} \cap S$, we want $m \ge 0$ and $m < 1$.

So all lines $L_{-m,2m} \cap S$, with $0 \le m < 1$, are parallel to $L_{-1,2} \cap S$, as well as the line $L_{1,-2} \cap S$.

8. In the Missing Strip plane $(\mathcal{S}, \mathcal{L})$, for the line $\ell = L_{m,b}$, define

$$g_\ell(x,y) = \begin{cases} f_\ell(x,y) & \text{if } x < 0, \\ f_\ell(x,y) - \sqrt{1+m^2} & \text{if } x \geq 1. \end{cases}$$

Verify that $g_\ell : (\ell \cap \mathcal{S}) \to \mathbb{R}$ is a bijection.

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(8)

$$g_\ell(x,y) = \begin{cases} f_\ell(x,y) & \text{if } x < 0 \\ f_\ell(x,y) - \sqrt{1+m^2} & \text{if } x \geq 1. \end{cases}$$

In missing strip plane: $g_\ell : (\ell \cap \mathcal{S}) \to \mathbb{R}$ is a bijection.

Pf: For 1-1, say $A = (x_1, y_1)$ and $B = (x_2, y_2)$ and say $g_\ell(A) = g_\ell(B)$. We must show that $A = B$.

Cases (1) $x_1 < 0$, $x_2 < 0$

(2) $x_1 < 0$, $x_2 \geq 1$

(3) $x_1, x_2 \geq 1$.

Case (1) This means $f_\ell(A) = f_\ell(B)$

or $x_1 \sqrt{1+m^2} = x_2 \sqrt{1+m^2}$, so $x_1 = x_2$

Then $y_1 = y_2$ since A, B are both on $y = mx + b$.

Case (2) This means $f_\ell(A) = f_\ell((x_1, y_1)) = f_\ell((x_2, y_2)) - \sqrt{1+m^2}$

or $x_1 \sqrt{1+m^2} = \sqrt{1+m^2}(x_2 - 1)$

So $x_1 = x_2 - 1$ or $x_2 - x_1 = 1$.

/over

(8) cont'd

But $x_1 < 0$ and $x_2 \geq 1$ means $x_2 - x_1 > 1$, contr^{n}.

So this case can't happen.

Case (3)

Now $f_\ell((x_1,y_1)) - \sqrt{1+m^2} = f_\ell((x_2,y_2)) - \sqrt{1+m^2}$

so $x_1\sqrt{1+m^2} = x_2\sqrt{1+m^2}$, so $x_1 = x_2$,

and then $y_1 = mx_1 + b = mx_2 + b = y_2$. Hence g_ℓ is 1-1.

Now we verify that g_ℓ is onto $\mathbb{R}$:

Let $t \in \mathbb{R}$, and say ℓ is $y = mx + b$ (intersected with S).

Now $g_\ell((x,y)) = \begin{cases} x\sqrt{1+m^2} & \text{if } x < 0 \\ (x-1)\sqrt{1+m^2} & \text{if } x \geq 1 \end{cases}$

Suppose $t < 0$. Then let $x_0 = \dfrac{t}{\sqrt{1+m^2}}$ and $y_0 = mx_0 + b$.

Then $g_\ell((x_0,y_0)) = x_0\sqrt{1+m^2} = t$, as required.

Suppose $t \geq 0$. Then let $x_0 = \dfrac{t}{\sqrt{1+m^2}} + 1$ (≥ 1), & $y_0 = mx_0 + b$.

Then $g_\ell((x_0,y_0)) = (x_0 - 1)\sqrt{1+m^2} = \dfrac{t}{\sqrt{1+m^2}} \cdot \sqrt{1+m^2} = t$, as req'd.

Hence g_ℓ is onto $\mathbb{R}$. So g_ℓ is a bijection.

12. Given a triangle, $\triangle ABC$, in a metric geometry, and points D, E with $A—D—B$ and $C—E—B$, is it always the case that $\overleftrightarrow{AE} \cap \overleftrightarrow{CD} \neq \emptyset$?

Explain your answer carefully.
(Hint: Recall that the Missing Strip plane is not a Pasch geometry.)

SOLUTION:
Not so, in a metric geometry which isn't Pasch. An example in the Missing Strip plane suffices to show this. Take $A = (-1, 0)$, $B = (2, 0)$ and $C = (2, 3)$. Then take points $D = (-\frac{1}{2}, 0)$ with $A—D—B$, and $E = (2, 2)$ with $C—E—B$.

Recall that the *missing* points are $\{(x, y) \mid 0 \leqslant x < 1\}$.

Now the line joining D and C has slope $6/5$ and equation $y = \frac{6}{5}x + \frac{3}{5}$, while the line joining A and E has slope $2/3$ and equation $y = \frac{2}{3}x + \frac{2}{3}$. These lines must be taken to intersect the points of the plane. They would meet at $(\frac{1}{8}, \frac{3}{4})$, but this point is not in the Missing strip plane, so in this plane, $\overleftrightarrow{AE} \cap \overleftrightarrow{CD} = \emptyset$.